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# Explicit Analytic Solutions of Classical Scalar Field Cosmology

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A class of explicit and exact solutions is obtained for the equations governing the evolution of spatially flat FLRW spacetimes in interaction with a classical massive scalar field in the presence of conformal coupling and of a quartic self-interaction potential. These solutions only exist for nonvanishing cosmological constant. The equation of state is calculated self-consistently.

## 1. INTRODUCTION

Two years ago we started a study of the dynamical behavior of a spatially flat FLRW spacetime in interaction with a classical massive scalar field with conformal coupling to gravity.

As a first step, we applied new mathematical methods developed in our group for finding first integrals and constants of motion (invariants) of the coupled set of the Einstein and Klein–Gordon equations with a quartic self-interaction potential for the massive scalar field in the presence of a cosmological constant. This led us quite unexpectedly to find explicit analytical solutions in six cases corresponding to particular relationships between the physical parameters of the theory: the mass $m$ of the scalar field, the quartic self-interaction coupling constant $\Omega$, and the cosmological constant $\Lambda$. Among these solutions, as we show later in this article, some are of real physical interest. In spite of their interest these solutions are not generic,

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being limited by specific constraints on the parameters and to certain sets of initial conditions. However, a few months later we discovered a much stronger result: for any values of these parameters, the phase space of the above set of differential equations reduces to a two-dimensional manifold embedded in a three-dimensional phase space $\{\psi, \dot{\psi}, H\}$, where $\psi$ is the scalar field, $\dot{\psi}$ is its time derivative, and $H$ is the Hubble function. This theorem generically *excludes any chaotic behavior* for the cosmological histories of the spatially flat FLRW universe in the presence of this potential [1].

The power of that result was reinforced when we finally discovered that its validity is extended to any form of the self-interacting potential of the scalar field and to any value of the nonminimal coupling constant [2]. The only limitation of the no-chaos theorem was the spatial flatness of the FLRW universes considered.

The topology of the two-dimensional manifold to which the dynamics of these universes are restricted is rather complex. Depending on the values of parameters related to the mass of the scalar field, the functional form of the self-interaction potential and the cosmological constant, holes may appear in that manifold, corresponding to forbidden regions in the phase space. The phase portrait of the dynamical system is strongly influenced by the presence of these holes. The complex nature of the topology could be the cause of the apparent chaotic behavior observed in the numerical solutions for that system. Indeed, numerical integration introduces errors due to the discretization of the time derivative and to the approximation of real numbers by floating point numbers. The latter implies that initial conditions almost never (in the sense of measure) lie on the permitted manifold, whereas the discretization implies a drastic change in the dynamics which is replaced by a system of finite-difference nonlinear equations. Such equations generally have chaotic solutions that are highly sensitive to small variations in the initial conditions. Hence, from any initial point which is not exactly on the allowed two-dimensional manifold, albeit infinitesimally close to it, a solution will emerge whose distance from that manifold grows exponentially.

Our purpose in this article is not to present these generic properties, which have been the subject of detailed publications [1, 2], but rather to expose the more particular but explicit and analytic solutions obtained in the first part of our work. Although restricted to a specific form of the self-interaction potential, to the particular (conformal) value 1/6 of the nonminimal coupling constant between the scalar field and gravity, and to particular relations between the remaining parameters, we think that these exact solutions of the self-consistent cosmological equations are of physical relevance.

The article begins with a mathematical section describing a method for finding invariants based on a general theory of nonlinear ODEs that we have been developing over the past few years [3–14]. In the next section, we briefly present the self-consistent cosmological equations that we are solving and their context; finally, the six types of solutions obtained and their properties are described.

## 2. QUASI-MONOMIAL APPROACH TO ODEs AND INVARIANTS

### 2.1. General Framework

We give here a short presentation of a theory of ordinary differential equations (ODEs) that addresses the large class of equations that can be reduced to the standard form [3–14]

$$\dot{X}_i = X_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} X_k^{B_{jk}} \qquad (i = 1, \ldots, n) \tag{2.1}$$

where the numbers $A_{ij}$ and $B_{jk}$ are real or complex and constant. The symbol $\dot{X}_i$ denotes the time derivative of $X_i$. The integer $m$ is the number of monomials, or more precisely, of quasi-monomials (since the numbers $B_{jk}$ are not necessarily integers) $\Pi_{k=1}^{n} X_k^{B_{jk}}$ appearing in the system (2.1). In particular, the class of systems of type (2.1) includes all the equations with polynomial right-hand side. The factor $X_i$ appearing in the right-hand side of equation (2.1) is extracted from the general quasi-polynomial in order to define the logarithmic time derivative of $X_i(t)$. Systems of the form (2.1) are by no means restricted to (quasi-)polynomial nonlinearities. Indeed, it is shown below that most systems of ODEs with right-hand sides including more general functions can be exactly reduced to the form (2.1) by changes of variables and appropriate embeddings. Hence, the class of equations (2.1) involves most of the ODEs of interest in physics.

The form (2.1) represents a class of ODEs in a notation that highlights two mathematical objects: the rectangular matrices $A(n \times m)$ and $B(m \times n)$. That these objects are more than a mere notation appears through the invariance in form of Eq. (2.1) under the so-called quasi-monomial (QM) transformations

$$Y_i = \prod_{j=1}^{n} X_j^{C_{ij}} \tag{2.2}$$

where $C$ is an invertible $n \times n$ matrix. This property shown in the next paragraphs provides a decomposition of the set of equations (2.1) in equivalence classes. Moreover, as we will see, in each equivalence class a canonical representative exists in the form of a Lotka–Volterra (LV) equation [15]

$$\dot{Y}_i = Y_t \sum_{j=1}^{m} M_{ij} Y_j \qquad (i = 1, \ldots, m) \tag{2.3}$$

with

$$M = B \cdot A$$

This canonical LV equation is the basic tenet of many of the results obtained. Among these we were able to obtain an analytic expression for the general term of the Taylor expansion of the general solution $Y_i(t)$ for the initial condition problem of Eq. (2.3) [5]. The analytic form of the Poincaré normalizing series was also derived [13]; this series can then be transformed in terms of the original variables $x_i$ of Eq. (2.1) and also allows one to use resummation techniques, and hence leads to new and efficient methods of solving systems of nonlinear ODEs. In the following we do not detail these properties, but we rather focus on another category of results obtained via Eq. (2.3). The latter are methods for finding quasi-polynomial invariants or Lie symmetries, the common feature of these results being the reduction of the dimension of the system of ODEs [8, 9].

Before entering into the details of the quasi-monomial theory of ODEs we discuss general aspects of the notion of integrability. This is a rather elusive notion associated with the possibility of obtaining closed-form solutions for a given system of ODEs or, more generally, of reducing the dimension of that system. Integrability is also associated with the notion of "regularity" of the solutions in the sense that these are monotonic, periodic, or quasi-periodic functions of time. This excludes chaotic solutions which are more irregular functions of time and which possess the property of instability with respect to small variations of the initial conditions.

More precisely, a dynamical system of dimension $n$, i.e., a set of $n$ autonomous ordinary differential equations of the first order

$$\dot{x}_i = f_i(x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n) \tag{2.4}$$

is said to be *integrable* if it possesses $n - 1$ analytic constants of motion or invariants. These are functions $I_i(x_i, \ldots, x_n)$ of the dependent variables $X_i(t)$ that remain constant along any trajectory which is a solution of the system (2.4). Each of these invariants allows for the elimination of one dependent variable in terms of the $n - 1$ remaining ones. Hence, if $n - 1$ such functions $I_i$ are known, one obtains a system

$$\begin{aligned} I_1(x_1(t), \ldots, x_n(t)) &= C_i \\ &\vdots \\ I_{n-1}\,(x_1(t), \ldots, x_n(t)) &= C_{n-1} \end{aligned} \tag{2.5}$$

where the constants $C_i$ are given by

$$\begin{aligned} C_1 &\equiv I_1(x_1(0), \ldots, x_n(0)) \\ &\vdots \\ C_{n-1} &\equiv I_{n-1}(x_1(0), \ldots, x_n(0)) \end{aligned} \tag{2.6}$$

Thus, if the system (2.5) is invertible, one can express $n - 1$ of the $x_i$ in terms of one of them; this invertibility property implies of course certain regularity properties of the functions $I_i$. The reduction leads then to only one differential equation, which in turn can be integrated by quadrature.

The above considerations could appear at first sight as a clear definition of integrability; however, some caution should be taken. Indeed, when an algorithm for finding such invariants is designed, the set of functions in terms of which these invariants are constructed has to be made precise. One can look, e.g., for polynomial invariants, or one may try to find invariants in a larger class of functions, such as the rational or the analytic functions of $x_1, \ldots, x_n$. In fact, one may look for invariants in successive algebraic extensions of the polynomial functional set; this raises two questions. The first one is about the nature of the maximal algebraic extension of the function ring in which to look for invariants and about its relationship with the chaotic or regular nature of the solutions. The second question is a decidability issue: a given dynamical system may admit invariants of various functional forms. Hence, the fact that a system does not possess any invariant of a given form does not mean that one can decide whether it is integrable or chaotic.

An example is the well-known Lotka–Volterra predator–prey system

$$\begin{aligned} \dot{x} &= \alpha x - \beta xy \\ \dot{y} &= -\gamma y + \delta xy \end{aligned} \tag{2.7}$$

where $\alpha$, $\beta$, $\gamma$, and $\delta$ are positive real constants. For most values of these parameters, this system does not admit any polynomial or quasi-polynomial invariant. However, for any value of the parameters, it admits an invariant belonging to the Liouville ring of functions (which include rational functions and logarithms),

$$I(x, y) = \gamma \log x - \delta x + \alpha \log y - \beta y \tag{2.8}$$

That this is an invariant can easily be checked by computing $dI/dt$ modulo the system (2.7).

This example clearly shows that without knowing the maximal extension ring of functions in which to look for invariants of a given system, no finite algorithm can be constructed in order to decide whether that system is integrable. Although algorithms such as the Painlevé test [16, 17] exist, they are limited to the status of mere conjectures.

## 2.2. Quasi-Polynomial Equations

Let us define the class of quasi-polynomial differential equations on $\mathbb{R}^n$ or $\mathbb{C}^n$ as

$$\dot{x}_i = x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}} \qquad (i = 1, \ldots, n) \tag{2.9}$$

$A$ and $B$ in Eq. (2.9) are real or complex constant rectangular matrices. The number $m$ is related to the number of quasi-monomials appearing in the vector field of Eq. (2.9), and generally $m$ is different from $n$.

As said above, many systems which are not in the quasi-polynomial form (2.9) can be recast in this form; the subject was studied exhaustively in refs. 8 and 9. Here, we present simple examples in order to clarify the procedure that reduces non-quasi-polynomial systems to the quasi-polynomial form.

*Example 1* (Van der Pol and Duffing equations). Let us consider the equation

$$\ddot{x} - \alpha\dot{x} - \beta x^2\dot{x} - \gamma x - \delta x^3 = \mathscr{E}\cos(t) \tag{2.10}$$

which includes the cases of the forced Van der Pol and Duffing oscillators. First, we transform Eq. (2.10) into a first-order system; we define new variables $x_1 = x$ and $x_2 = \dot{x}$ and obtain

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_2 + \beta x_1^2 x_2 + \gamma x_1 + \delta x_1^3 + \mathscr{E}\cos(t) \end{aligned} \tag{2.11}$$

In order to have an autonomous system, one introduces the additional variables $x_3 = \cos(t)$ and $x_4 = \sin(t)$, obtaining the quasi-polynomial system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_2 + \beta x_1^2\, x_2 + \gamma x_1 + \delta x_1^3 + \mathscr{E} x_3 \\ \dot{x}_3 &= -x_4 \\ \dot{x}_4 &= x_3 \end{aligned} \tag{2.12}$$

The system is finally recast in the form (2.9),

$$\begin{aligned} \dot{x}_1 &= x_1(x_2 x_1^{-1}) \\ \dot{x}_2 &= x_2(\alpha + \beta x_1^2 + \gamma x_1 x_2^{-1} + \delta x_1^3 x_2^{-1} + \mathscr{E} x_3 x_2^{-1}) \\ \dot{x}_3 &= x_3(-x_4 x_3^{-1}) \end{aligned}$$

$$\dot{x}_4 = x_4(x_3 x_4^{-1}) \tag{2.13}$$

By ordering the quasi-monomials which appear on the right-hand side as

$$\begin{gathered} y_1 = 1, \qquad y_2 = x_2 x_1^{-1}, \qquad y_3 = x_1{}^2 \\ y_4 = x_1\, x_2^{-1}, \qquad y_5 = x_1{}^3 x_2^{-1}, \qquad y_6 = x_3 x_2^{-1} \\ y_7 = x_4 x_3^{-1}, \qquad y_8 = x_3 x_4^{-1} \end{gathered} \tag{2.14}$$

we obtain the matrices $B$ and $A$ defined in Eq. (2.9),

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \tag{2.15}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & \beta & \gamma & \delta & \mathscr{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The case in which there is no external periodic force is given by Eq. (2.11) with $\mathscr{E} = 0$. Then, if we consider the order of the quasi-monomials in (2.14), we have

$$B = \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ 2 & 0 \\ 1 & -1 \\ 3 & -1 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \alpha & 0 & \beta & \gamma & \delta \end{pmatrix} \tag{2.16}$$

*Example 2.* As a second example, let us consider a model which describes the oscillation in time of the concentration of electron-hole pairs ($x_1$) and excitons ($x_2$) in an intrinsic semiconductor. If the higher order kinetics is allowed in the model, the process is described by the equations

$$\begin{aligned} \dot{x}_1 &= g - c x_1{}^2 x_2 \\ \dot{x}_2 &= c x_1{}^2 x_2 - \frac{k x_2}{(1 + q x_2)^m} \end{aligned} \tag{2.17}$$

where $g$, $c$, and $k$ are defined in the process consisting of the following steps:

- *Photogeneration of carriers*: $\gamma \xrightarrow{g} e + h$
- *Stimulated production of excitons*: $e + h + x_2 \xrightarrow{c} 2x_2$
- *Radiative decay of excitons*: $x_2 \xrightarrow{k} \gamma$

The system (2.16) can be put in the quasi-polynomial form (2.9) if we introduce the variable $x_3 = (1 + qx_2)^{-1}$:

$$\begin{aligned} \dot{x}_1 &= x_1(gx_1^{-1} - cx_1x_2) \\ \dot{x}_2 &= x_2(cx_1{}^2 - kx_3{}^m) \\ \dot{x}_3 &= x_3(-cqx_1{}^2x_2x_3 + qkx_2x_3^{m+1}) \end{aligned} \tag{2.18}$$

The matrices $B$ and $A$ are then

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & m \\ 2 & 1 & 1 \\ 0 & 1 & m+1 \end{pmatrix}, \qquad A = \begin{pmatrix} g & -c & 0 & 0 & 0 & 0 \\ 0 & 0 & c & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & -cq & qk \end{pmatrix} \tag{2.19}$$

More generally, a systematic method for recasting a general class of differential equations into the quasi-polynomial form is given by the following proposition [8, 9].

*Proposition 2.1.* Let us consider a dynamical system of the general form

$$\dot{x}_i = \sum_{j=1}^{N_i} D_j \prod_{k=1}^{n} x_k^{E_{jk}} \prod_{k=1}^{p} f_k(x)^{F_{jk}} \qquad (i = 1, \ldots, n)$$

where $E_{jk}, F_{jk}, D_j \in \mathbb{R}$ and $f_k: \mathbb{R}^n \to \mathbb{R}$ are functions with partial derivatives which can be expressed in the form

$$\frac{\partial f_l}{\partial x_i} = \sum_{j=1}^{M_i} G_j^{il} \prod_{k=1}^{n} x_k^{H_{jk}} \prod_{k=1}^{q} f_k(x)^{K_{jk}} \qquad (i = 1, \ldots, n;\ l = 1, \ldots, p)$$

with $G_j^{il}, H_{jk}, K_{jk} \in \mathbb{R}$. Let us consider the variables

$$y_{rs} = f_r^{a_{rs}} \prod_{k=1}^{n} x_k^{b_{rsk}a_{rs}} \neq 0 \qquad (s = 1, \ldots, l_r)$$

Then the dynamical system of equations for the variables $x_i$ and $y_{rs}$ is a quasi-polynomial dynamical system.

By using this proposition one can see that many different quasi-polynomial representations can exist for a given dynamical system. A relevant question is: what is the quasipolynomial representation with the smallest number of quasi-monomials? In refs. 8 and 9 this issue is analyzed exhaustively and criteria are given for minimal recasting in the quasi-polynomial form.

## 2.3. Quasi-Monomial Transformation

Let us consider the quasi-polynomial system (2.9) and a transformation of the form

$$y_i = \prod_{j=1}^{n} x_j^{C_{ik}} \qquad (i = 1, \ldots, n) \tag{2.20}$$

where $C$ is a non-singular matrix. The transformations of the form (2.20) are called quasi-monomial transformations (QMTs) and have been studied by many authors in different contexts [3–6]. The inverse of the QMT in (2.20) is also a QMT given by the inverse matrix of $C$ as

$$x_i = \prod_{j=1}^{n} y_j^{C_{ik}^{-1}} \qquad (i = 1, \ldots, n) \tag{2.21}$$

The most important property of QMTs may be synthesized in the following proposition:

*Proposition 2.2.* Let us consider a QMT given by (2.20); the QP-system (2.9) is transformed into another QP-system given by

$$\dot{y}_i = y_i \sum_{j=1}^{m} A'_{ij} \prod_{k=1}^{n} y_k^{B'_{jk}} \tag{2.22}$$

with

$$A' = C^{-1}A, \qquad B' = BC \tag{2.23}$$

This proposition asserts that, for nonlinear ordinary differential equations, the representation (2.9) is covariant under QMTs. It is easy to see that the matrix $M = B'A' = BA$ is invariant under QMTs.

Let the rank of the matrix $B$ in (2.9) be $p$. Since rank($B$) $= p \leq n, m$, there are $p$ linearly independent row or column vectors in the matrix $B$. Let us consider $N_1, \ldots, N_{n-p}$ as a column vector basis for the kernel of $B$. We can find $p$ column vectors $L_1, \ldots, L_p$ such that $L_1, \ldots, L_p, N_1, \ldots, N_{n-p}$ are linearly independent. Thus we define the invertible matrix $C$ as

$$C = (L_1 \dots L_p N_1 \dots N_{n-p}), \qquad N_1, \dots, N_{n-p} \in \text{kernel}(B) \quad (2.24)$$

The invertible matrix $C$ defines a QMT of the form (2.20). The system (2.9) is transformed into a system (2.22) with the new matrices $A'$ and $B'$ as in (2.23). The matrix $B'$ is

$$\begin{aligned} B' &= BC = (BC_1 \dots BC_n) \\ &= (BL_1 \dots BL_p BN_1 \dots BN_{n-p}) \\ &= (B'_1 \dots B'_1\ 0 \dots 0) \end{aligned}$$

Finally, one has

$$B' = \left(\begin{array}{ccc|ccc} B'_{11} & \cdots & B'_{1p} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B'_{m1} & \cdots & B'_{mp} & 0 & \cdots & 0 \end{array}\right) \quad (2.25)$$

As a consequence, the system (2.22) is decoupled as follows:

$$\begin{aligned} \dot{y}_1 &= y_1 \sum_{j=1}^{m} A'_{1j} \prod_{k=1}^{p} y_k^{B'_{jk}} \\ &\vdots \\ \dot{y}_p &= y_p \sum_{j=1}^{m} A'_{pj} \prod_{k=1}^{p} y_k^{B'_{jk}} \\ &\text{- - - - - - - - - - - - - - - - - - - -} \\ \dot{y}_{p+1} &= y_{p+1} \sum_{j=1}^{m} A'_{p+1,j} \prod_{k=1}^{p} y_k^{B'_{jk}} \\ &\vdots \\ \dot{y}_n &= y_n \sum_{j=1}^{m} A'_{nj} \prod_{k=1}^{p} y_k^{B'_{jk}} \end{aligned} \quad (2.26)$$

Recall that $p \leq m$ and $\text{rank}(B) = p$; this result, obtained by Brenig and Goriely [4], permits one to conclude that in the study of quasi-polynomial systems it is only necessary to consider the QP-systems (2.9) with $m \geq n$ and $\text{rank}(B) = n$.

### 2.4. Lotka–Volterra Systems

It is known [3–6] that any QP-system (2.9) is related to a quadratic homogeneous Lotka–Volterra system (LV system). A first step in establishing this connection consists in embedding the system (2.9) into an appropriate manifold. By reasoning as in the previous section and considering $m \geq n$ and rank$(B) = n$, we introduce $m - n$ variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and obtain a new system defined on $\mathbb{R}^m$:

$$\begin{aligned} \dot{x}_i &= x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}} \qquad (i = 1, \ldots, n) \\ \dot{x}_p &= 0 \qquad\qquad (p = n+1, \ldots, m) \end{aligned} \tag{2.27}$$

with initial conditions $x_p(0) = 0$. There is actually no dynamics for the new variables $x_{n+1}, x_{n+2}, \ldots, x_m$. We recast the system (2.27) as

$$\dot{x}_i = x_i \sum_{j=1}^{m} \mathcal{A}_{ij} \prod_{k=1}^{m} x_k^{\mathcal{B}_{jk}} \qquad (i = 1, \ldots, m) \tag{2.28}$$

where

$$\mathcal{A} = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \\ \hline 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \qquad \mathcal{B} = \left(\begin{array}{ccc|ccc} B_{11} & \cdots & B_{1n} & b_{11} & \cdots & b_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mn} & b_{m1} & \cdots & b_{mr} \end{array}\right) \tag{2.29}$$

where $\mathcal{A}$ and $\mathcal{B}$ are $m \times m$ square matrices and $r = m - n$. It is clear that there exist $rm$ free parameters $b_{ij}$ in the matrix $\mathcal{B}$. Since rank$(B) = n$, one can always choose $b_{ij}$ such that $\det(\mathcal{B}) \neq 0$, and define a QMT given by

$$y_j = \prod_{k=1}^{m} x_k^{\mathcal{B}^{-1}_{jk}} \qquad (j = 1, \ldots, m) \tag{2.30}$$

Proposition 2.2 allows one to conclude that the dynamics obeyed by the variables $y_j$ are governed by

$$\dot{y}_j = y_j \sum_{k=1}^{m} M_{jk} y_j \tag{2.31}$$

with $M = \mathcal{B}\mathcal{A} = BA$. This system is the so-called quadratic homogeneous Lotka–Volterra system [15] and describes the dynamics of the quasi-monomials appearing in Eq. (2.9), which are in this context the basic variables for the system's dynamics. The reasoning in the above paragraph permits us to assert:

*Proposition 2.3.* There exist an embedding and a QMT which transform any QP-system (2.9), determined by the matrices $A$ and $B$, into a homogeneous Lotka–Volterra system (2.31) defined by the matrix $M = BA$.

*Examples.* As an example, we consider the quasi-polynomial representation of the well-known Lorenz system

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2 \end{aligned} \tag{2.32}$$

The latter may be recast in the form (2.9) with the quasi-monomials

$$\begin{gathered} y_1 = x_1{}^0 x_2{}^0 x_3{}^0, \qquad y_2 = x_1^{-1} x_2{}^1 x_3{}^0, \qquad y_3 = x_1{}^1 x_2^{-1} x_3{}^0 \\ y_4 = x_1{}^1 x_2^{-1} x_3{}^1, \qquad y_5 = x_1{}^1 x_2{}^1 x_3^{-1} \end{gathered} \tag{2.33}$$

The matrices $A$ and $B$ are

$$A = \begin{pmatrix} -\sigma & \sigma & 0 & 0 & 0 \\ -1 & 0 & \rho & -1 & 0 \\ -\beta & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.34}$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \tag{2.35}$$

and the $M$ matrix of the corresponding Lotka–Volterra system (2.31) is given by

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sigma - 1 & -\sigma & \rho & -1 & 0 \\ 1 - \sigma & \sigma & -\rho & 1 & 0 \\ 1 - \sigma - \beta & \sigma & -\rho & 1 & 1 \\ \beta - 1 - \sigma & \sigma & \rho & -1 & -1 \end{bmatrix} \tag{2.36}$$

The dimension of $M$ is given by the number of quasi-monomials in (2.32).

As a second example, let us consider the forced oscillator system given by Eq. (2.13) and the matrices $A$ and $B$ defined in (2.15); its associated LV system is given by the matrix $M = BA$, which we write explicitly as

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & -1 & \beta & \gamma & \delta & \mathcal{E} & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha & 1 & -\beta & -\gamma & -\delta & -\mathcal{E} & 0 & 0 \\ -\alpha & 3 & -\beta & -\gamma & -\delta & -\mathcal{E} & 0 & 0 \\ -\alpha & 0 & -\beta & -\gamma & -\delta & -\mathcal{E} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix} \tag{2.37}$$

The Lotka–Volterra matrix $M$ corresponding to the semiconductor model (2.18) with the matrices $B$ and $A$ in (2.19) is

$$M = \begin{pmatrix} -g & c & 0 & 0 & 0 & 0 \\ g & -c & c & -k & 0 & 0 \\ 2g & -2c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -mcq & mqk \\ 2g & -2c & c & -k & -cq & qk \\ g & -c & c & -k & -(m+1)cq & (m+1)qk \end{pmatrix} \tag{2.38}$$

## 2.5. Differential Operators and Invariant Surfaces

In this part, we are essentially interested in the study of invariant surfaces of the quasi-polynomial dynamical systems (2.9). This area constitutes an excellent illustration of the powerful algebraic tools that can be introduced in nonlinear dynamics through the quasi-monomial approach. The results below are detailed in refs. 10 and 11.

*Definition 2.1 (Invariant Surfaces).* Let us consider the flow $X(x_0, t)$ of the QP-system (2.9). A surfaces $S \subset \mathbb{R}^n$ is an invariant surface of the dynamical system (2.9) if

$$\forall x_0 \in S \Rightarrow X(x_0, t) \in S \quad \forall t$$

Recall that the flow $X(x_0, t)$ is the solution of the system (2.9) with initial condition $x_0$, i.e., $X(x_0, 0) = x_0$.

To study the invariant surfaces, it is interesting to associate to a QP-system (2.9) the differential operator

$$\delta_{(A,B)} = \sum_{i=1}^{n} (Ax^B)_i \, x_i \frac{\partial}{\partial x_i} \tag{2.39}$$

where

$$(Ax^B)_i = \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}$$

In fact, if we consider the set of differentiable functions defined as

$$\mathbb{R}^{n\mathbb{R}} = \{ f\colon \mathbb{R}^n \to \mathbb{R} | \, f \text{ is differentiable}\} \tag{2.40}$$

then the operator $\delta_{(A,B)}$ is a map from $\mathbb{R}^{n\mathbb{R}}$ to $\mathbb{R}^{n\mathbb{R}}$. The embedding (2.27) maps the invariant surface of the QP-system (2.9) into another invariant surface of the QP-system (2.28). Moreover, a QMT transforms invariant surfaces into invariant surfaces. Thus, in order to study the invariant surfaces, it is only necessary to examine the invariant surfaces of the differential operators which belong to the set

$$Q\,Der(x) = \{\delta_{(A,B)} | A, B \text{ are square } m \times m \text{ matrices, } \det(B) \neq 0\} \tag{2.41}$$

When $A = M$ and $B = I$ we have a homogeneous Lotka–Volterra differential operator given by

$$\delta_M = \sum_{i=1}^{m} (My)_i \, y_i \frac{\partial}{\partial y_i} \tag{2.42}$$

which corresponds to the system of ordinary differential equations (2.31). The study of invariant surfaces gives important information on the topology of the orbits representing the solutions in the phase space of the system.

## 2.6. Quasi-Polynomial Invariants

Let us consider a quasi-polynomial function on $\mathbb{R}^n$ given by

$$F = \sum_{i=1}^{N} F_i \prod_{k=1}^{n} x_k^{C_{ik}} \tag{2.43}$$

where $C_{ik} \in \mathbb{R}$. The QMT in (2.20) transforms $F$ into

$$F = \sum_{i=1}^{N} F_i \prod_{k=1}^{n} y_k^{[C\mathcal{B}^{-1}]_{ik}} \tag{2.44}$$

Hence, we can conclude that $F$ is also a quasi-polynomial function in the new variable $y_i$. We summarize this property in the following proposition:

*Proposition 2.4.* The QMT (2.20) transforms a quasi-polynomial function (2.43) into another quasi-polynomial function given by (2.44) with the same number of quasi-monomial terms.

Now, if we consider a QMT, we conclude that

$$\delta_M(F) = 0 \Leftrightarrow \delta_{(A,B)}(F) = 0 \tag{2.45}$$

Proposition 2.4 and the relation (2.45) permit one to assert that, in order to study the class of quasi-polynomial invariants (QP-invariants), it is only necessary to analyze the invariants of the LV system. Therefore, in the rest of this section we study the QP-invariants of the LV system.

Let us consider a QP-invariant of the form

$$F = \sum_{i=1}^{N} F_i y^{\beta_i} \tag{2.46}$$

with $y^{\beta_i} = \Pi_{k=1}^{m} y_k^{\beta_{ik}}$ and $\beta_i = [\beta_{i1}, \ldots, \beta_{im}]$; the function $F$ satisfies the equation

$$\delta_M(F(y)) = 0 \tag{2.47}$$

One obtains

$$\sum_{i=1}^{N} F_i y^{\beta_i} \sum_{j=1}^{m} (\beta_i M)_j y_j = 0$$

$$\sum_{i=1}^{N} \sum_{j=1}^{m} F_i [\beta_i M]_j y^{\beta_i + e_j} = 0 \tag{2.48}$$

where $[\beta_i M]_j = \Sigma_{k=1}^{m} \beta_{ik} M_{kj}$ and $e_j$ is a unit vector with the $j$th component equal to unity and with the remaining components vanishing.

Let us consider two monomials $y^{\beta_i}$ and $y^{\beta_j}$ in the set $S = \{y^{\beta_1}, \ldots, y^{\beta_N}\}$. They are said to be *connected* ($y^{\beta_i} \leftrightarrow y^{\beta_j}$) if $\beta_i - \beta_j \in Z^m$, where

$$Z^m = \{(z_i, \ldots, z_m) | z_i \in Z\}$$

The connection $\leftrightarrow$ is an equivalence relation on $S$; then the set $S$ is decomposed into the disjoint equivalence classes of this relation, and the number of equivalence classes in $S$ is different from $N$. We denote the different equivalence classes in this set by $[y^{\alpha_i}]$, where $i = 1, \ldots, r \le N$ and $\alpha_i = \beta_{q_i}$ with $q_i \in \{1, 2, \ldots, N\}$ and $q_i \neq q_j$ for $i \neq j$. We can rewrite $F$ as

$$F = F_{(1)} + F_{(2)} + \ldots + F_{(r)} \tag{2.49}$$

where $F_{(i)}$ is a quasi-polynomial with all its monomials in the same equivalence class $[y^{\alpha_i}]$. We know that this set of equivalence class is given by

$$[y^{\alpha_i}] = \{y^{\alpha_i + \eta_{ik}} | k = 1, \ldots, N_i, \eta_{ik} \in Z^m\} \tag{2.50}$$

where $N_i$ is the number of elements of $S$ equivalent to $y^{\alpha_i}$. We can state the following proposition:

*Proposition 2.5.* Let us consider the QP-invariant $F$ (2.49); then the $F_{(i)}$ are also QP-invariants.

*Proof.* By performing the differentiation $\delta_M(F) = 0$, one obtains

$$\delta_M(F_{(1)}) + \ldots + \delta_M(F_{(r)}) = 0$$
$$f_{(1)} + \ldots + f_{(r)} = 0$$

where $f_{(i)} = \delta_M(F_{(i)})$. By using (2.48) and (2.50) one can see that all the quasi-monomials in $f_{(i)}$ are of the form $y^{\alpha_i+\eta_{ik}+e_p}$ ($p = 1, \ldots, m$). This implies that all the monomials in $f_{(i)}$ are in the equivalence class $[y^{\alpha_i}]$. Now, for $i \neq j$ ($[y^{\alpha_i}] \neq [y^{\alpha_j}]$), one can see that every quasi-monomial in $f_{(i)}$ is different from every quasi-monomial in $f_{(j)}$, or otherwise one would have a quasi-monomial such that $y^{\alpha_i+\eta_{ip}+e_r} = y^{\alpha_j+\eta_{jq}+e_s}$. Thus, we have $(\alpha_i - \alpha_j) \in Z^m$ and consequently $y^{\alpha_i} \leftrightarrow y^{\alpha_j}$ and $[y^{\alpha_i}] = [y^{\alpha_j}]$ for $i \neq j$. Finally, since all the monomials in $f_{(i)}$ and $f_{(j)}$ ($i \neq j$) are different, we can conclude that $f_{(i)}$ and $f_{(j)}$ are linearly independent. This implies that $f_{(i)} = \delta_M(F_{(i)}) = 0$. ∎

The importance of this result can be understood if one rewrites $F_{(i)}$ by using (2.50),

$$F_{(i)} = y^{\alpha_i} \sum_{k=1}^{N_i} F_{ik} y^{\eta_{ik}} \qquad (\eta_{ik} \in Z^m) \tag{2.51}$$

We factorize (2.51) by the highest common multiple $y^{\vartheta_i}$ for the set of quasi-monomials $y^{\eta_{ik}}$ to obtain

$$F_{(i)} = y^{(\alpha_i-\vartheta_i)} \sum_{k=1}^{N_i} F_{ik} y^{(\eta_{ik}+\vartheta_i)} \tag{2.52}$$

We have actually proven that $F_{(i)} = y^{\theta_i}\mathcal{P}_i(y)$, where $\mathcal{P}_i(y)$ is a polynomial function and $\theta_i \in \mathbb{R}^m$. Finally, one can conclude by virtue of Proposition 2.5 that a QP-invariant (2.46) can be decomposed in the form

$$F(y) = \sum_{i=1}^{r} y^{\theta_i}\mathcal{P}_i(y) \tag{2.53}$$

where $\mathcal{P}_i(y)$ is a homogeneous polynomial function. At this point we remember that if the number $N_i$ of monomials in the equivalence class $[y^{\alpha_i}]$ is one, then $F_{(i)}$ is a quasi-monomial invariant (QM-invariant). If $F_{(i)}$ has at least two quasi-monomials, then $\mathcal{P}_i(y)$ is a nonconstant polynomial. The QM-invariants have been exhaustively studied by Brenig and Goriely [4, 5] and Gouzé [6].

The relation $\delta_M(y^{\theta_i}\mathcal{P}_i(y)) = 0$ yields

$$\mathcal{P}_i \delta_M(y^{\theta_i}) + y^{\theta_i}\delta_M(\mathcal{P}_i) = 0$$
$$y^{\theta_i}[\mathcal{P}_i \textstyle\sum_{j=1}^m (\theta_i M)_j y_j + \delta_M(\mathcal{P}_i)] = 0 \tag{2.54}$$

and finally

$$\delta_M(\mathcal{P}_i) = \left(-\sum_{j=1}^{m} (\theta_i M)_j y_j\right)\mathcal{P}_i \tag{2.55}$$

The polynomial $\mathcal{P}$ which satisfies $\delta_M(\mathcal{P}) = \lambda\mathcal{P}$, where $\lambda = \sum_{j=1}^m \lambda_i y_i$ is a linear function, is called a *semi-invariant* or *Darboux polynomial* of the derivative $\delta_M$. In this work, $\lambda$ is called the *eigenvalue* of the semi-invariant. The following properties about semi-invariants of homogeneous derivatives are known in the literature [18]:

*Proposition 2.6.* Let us consider a semi-invariant $\mathcal{P}$ and its respective eigenvalue $\lambda$. $\mathcal{P}$ can be written as $\mathcal{P} = \prod_k \mathcal{P}_k^{r_k}$ $(r_k \in \mathbb{N})$, factorized in its irreducible factors $\mathcal{P}_k$. Thus $\mathcal{P}_k$ is a semi-invariant with eigenvalue $\lambda_k$ and $\lambda = \sum_k r_k\lambda_k$.

*Proposition 2.7.* Let us suppose that a semi-invariant $\mathcal{P}$ with eigenvalue $\lambda$ is written as $\mathcal{P} = \mathcal{P}_1 + \ldots + \mathcal{P}_s$, where $\mathcal{P}_i$ is a homogeneous polynomial with degree $n_i$ and $n_i \neq n_j$ for $i \neq j$. Then $\mathcal{P}_i$ is a semi-invariant with eigenvalue $\lambda$.

To conclude this section, note that, upon use of (2.53), (2.55), and Propositions 2.6 and 2.7, we can state the most important result of this section:

*Proposition 2.8 (Decomposition Theorem).* Any QP-invariant $F$ given by (2.46) can be decomposed in the form

$$F(y) = \sum_{i=1}^{r} y^{\theta_i}\mathcal{P}_i(y) \tag{2.56}$$

where $y^{\theta_i}\mathcal{P}_i(y)$ is a QP-invariant, $\mathcal{P}_i(y)$ is a homogeneous semi-invariant with eigenvalue given by (2.55), and $\mathcal{P}_i(y)$ can be decomposed in irreducible homogeneous semi-invariants.

## 2.7. The Computational Algorithm and Its Applications

Let us consider a class of invariants of the form

$$F = \prod_{i=1}^{m} y_i^{\alpha_i}\mathcal{P}(y_i, \ldots, y_m) \tag{2.57}$$

where $\alpha_i$ are real numbers and $\mathcal{P}(y)$ is a polynomial function. The equation for an invariant is

$$\dot{K} = \sum_{j=1}^{m} \frac{\partial K}{\partial y_i} \dot{y}_j = 0 \tag{2.58}$$

The use of (2.57) and of the LV form in (2.58) yields

$$\prod_{i=1}^{m} y_i^{\alpha_i}(\dot{\mathcal{P}} - \lambda\mathcal{P}) = 0 \tag{2.59}$$

where $\lambda = -\Sigma_{j=1}^{m} (\alpha M)_j U_j$ and $(\alpha M)_j = \Sigma_{k=1}^{m} \alpha_k M_{kj}$. The condition (2.59) is equivalent to $\dot{\mathcal{P}} = \lambda\mathcal{P}$. Thus, $\mathcal{P}$ is a semi-invariant.

We define the set of linear functions from $\mathbb{R}^m$ to $\mathbb{R}$ as

$$L[y] = \left\{ f: \mathbb{R}^m \to \mathbb{R} \middle| f = \sum_{j=1}^{m} f_j y_j,\ y \in \mathbb{R}^m \right\} \tag{2.60}$$

and the set of semi-invariants associated to the differentiation $\delta_M$

$$SI[y] = \{ f: \mathbb{R}^m \to \mathbb{R} | f \text{ is semi-invariant of } \delta_M \} \tag{2.61}$$

Let us consider the map $\overline{M}$: $R^m \to L[y]$ which associates a vector $\alpha \in R^n$ to a linear function $\Sigma_{i=1}^{m} (\alpha M)_i y_i$, and the map $\Lambda$: $SI[y] \to L[y]$ associating to any $\mathcal{P} \in SI[y]$ such that $\dot{\mathcal{P}} = \lambda\mathcal{P}$, its eigenvalue $\lambda$, i.e., $\Lambda(\mathcal{P}) = \lambda$. With these definitions, we conclude that a function of the type (2.57) is an invariant of the system if two conditions are satisfied:

(a) There is a semi-invariant for the system.
(b) *Image*$(\overline{M}) \cap$ *Image*$(\Lambda) \neq \emptyset$.

This method and its application are developed in refs. 10 and 11; a computer package with the implementation of this method is described in ref. 12.

## 3. FLRW COSMOLOGY WITH A SELF-INTERACTING SCALAR FIELD

In this section we present the Einstein equations for a spatially flat FLRW cosmological model with a self-interacting scalar field conformally coupled to gravity. The action for the model is

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left( \frac{-R[g]}{\kappa} + g^{\mu\nu}\psi_\mu\psi_\nu - m^2\psi^2 + \frac{\Omega\psi^4}{2} + \frac{R\psi^2}{6} + 18\Lambda \right) \tag{3.1}$$

where $\psi_\mu \equiv \partial\psi/\partial x^\mu$, $\Lambda$ is the cosmological constant (rescaled for economy of notation), $\kappa \equiv 8\pi G_N$ (where $G_N$ is Newton's constant), and $m$ is the scalar field mass. Notice the nonminimal (conformal) coupling of the field $\psi$ to the scalar curvature $R$ given by the term $R\psi^2/6$. Physical reasons for the introduction of this term are discussed in refs. 19 and 20. By using the FLRW metric with flat spatial sections

$$ds^2 = d\tau^2 - a^2(\tau)(dx^2 + dy^2 + dz^2) \tag{3.2}$$

we obtain the Einstein equations derived from the variation of the action (3.1),

$$\psi_{\tau\tau} + 3H\psi_\tau + m^2\psi - \Omega\psi^3 - \frac{R\psi}{6} = 0 \tag{3.3}$$

$$R = -6(H_\tau + 2H^2) = \frac{36\omega}{\kappa} - \kappa(\sigma - 3p) \tag{3.4}$$

$$3H^2 + \frac{9\omega}{\kappa} = \kappa\sigma \tag{3.5}$$

where $\omega \equiv \kappa^2\Lambda$, the Hubble constant is defined by

$$H = a_\tau/a \tag{3.6}$$

and the energy density $\sigma$ and pressure $p$ are, respectively,

$$\sigma = \tfrac{1}{2}[\psi_\tau^2 + H^2\psi^2 + H\partial_\tau(\psi^2) + m^2\psi^2 - \tfrac{1}{2}\Omega\psi^4] \tag{3.7}$$

$$p = \tfrac{1}{6}[\psi_\tau^2 + H^2\psi^2 + H\partial_\tau(\psi^2) - m^2\psi^2 - \tfrac{1}{2}\Omega\psi^4] \tag{3.8}$$

In refs. 1 and 2 it is shown that Eqs. (3.3)–(3.5) are equivalent to the two-dimensional system

$$\psi_\tau = -H\psi \pm \frac{1}{2\kappa}\sqrt{G(H,\psi)} \tag{3.9}$$

$$H_\tau = -\frac{6\omega}{\kappa} - 2H^2 + \alpha\psi^2 \tag{3.10}$$

where $\alpha \equiv \kappa m^2/6$ and

$$G(H, \psi) \equiv 72\omega + 24\kappa H^2 - 24\alpha\kappa\psi^2 + 2\kappa^2\Omega\psi^4 \tag{3.11}$$

The dimensional reduction of the equations of motion yields the no-chaos theorem referred to in Section 1.

The system (3.3)–(3.5) can be rewritten as a Hamiltonian system by employing the rescaled variables

$$\phi \equiv \sqrt{\frac{6}{\kappa}}\, a \tag{3.12}$$

$$\Psi \equiv \sqrt{\frac{\kappa}{6}}\, \phi\psi \tag{3.13}$$

and conformal time $t$ defined by

$$d\tau = a\, dt \tag{3.14}$$

instead of proper time $\tau$. The rescaled variables obey the equations

$$\Psi_{tt} = -\alpha\Psi\phi^2 + \Omega\Psi^3 \tag{3.15}$$

$$\phi_{tt} = \alpha\Psi^2\phi - \omega\phi^3 \tag{3.16}$$

which can also be derived from the Hamiltonian

$$E \equiv \frac{\Psi_t^2}{2} - \frac{\phi_t^2}{2} + \frac{\alpha}{2}\Psi^2\phi^2 - \frac{\Omega}{4}\Phi^4 - \frac{\omega}{4}\phi^4 \tag{3.17}$$

together with the energy constraint

$$E = 0 \tag{3.18}$$

The advantage of using the form (3.15) and (3.16) of the Einstein–Klein–Gordon equations is that the latter allows one to use formal results known for Hamiltonian systems. In particular, for a system with two degrees of freedom the existence of a second constant of motion (invariant) besides the energy ensures integrability of the system, in the sense that it can be completely reduced to an action–angle representation such that the system is trivially solved by direct integration (even though the coordinate transformation is usually not easy to obtain).

## 4. INTEGRABLE CASES AND EXACT SOLUTIONS

The purpose of this section is to enumerate the integrable cases admitting QP-invariants of Eqs. (3.15) and (3.16) and to obtain explicit solutions for these cases. As explained in the previous section, integrability is guaranteed if a single constant of motion (in addition to the Hamiltonian) is known. The determination of analytical constants of motion of a given Hamiltonian system is usually complicated and, in most cases, they only exist for a restricted range of values of the parameters in the equations. These values can be determined by using the Painlevé analysis [16, 17, 21].

The analysis was performed in ref. 17 for a general class of systems of which the Hamiltonian (3.17) is a particular case; the results were recovered in our approach. The values of $\alpha$, $\omega$, and $\Omega$ for which the system is integrable, and the corresponding integrals of motion, were obtained (see Table I).

The system (3.15) and (3.16) can be reduced to the first-order system

$$\begin{aligned} d\Psi/dt &= \Psi_t \\ d\Psi_t/dt &= -\alpha\Psi\phi^2 + \Omega\Psi^3 \\ d\phi/dt &= \phi_t \\ d\phi_t/dt &= \alpha\Psi^2\phi - \omega\phi^3 \end{aligned} \tag{4.1}$$

In order to make progress beyond the work of refs. 17 and 21, we attempted to integrate completely Eqs. (4.1). To obtain the solutions corresponding to the integrable cases, one must obtain an additional invariant. Note that if one knows three constants of motion $C_i(\Psi, \phi, \Psi_t, \phi_t)$ ($i = 1, 2, 3$), then one can write three variables as functions of the fourth one.

Let us assume that $\Psi_t$, $\phi$, and $\phi_t$ only depend on $\Psi$; then Eq. (4.1) yields $\Psi_t = F(\Psi)$ and, if $F \neq 0$, the solution is reduced to a quadrature,

$$t - t_0 = \int_{\Psi(t_0)}^{\Psi(t)} \frac{d\Psi}{F(\Psi)} \tag{4.2}$$

The invariant $I$ in Table I is then used to eliminate one of the variables $\Psi$, $\phi$, $\Psi_t$, or $\phi_t$ in (4.1). Using Eqs. (3.7) and (3.8), we compute the energy density $\sigma(\tau)$, the pressure $p(\tau)$, and the equation of state $p = p(\sigma)$.

After elimination of one variable in terms of $I$ in Eq. (4.1), the resulting three-dimensional system depends on the value of $I$, which is now a new free parameter in the equations. We looked for a third invariant $J$ in the set of QP-functions by using the approach described in Section 2, and we were able to obtain a positive result only for $I = 0$, with the corresponding explicit solutions. The third invariant $J$ obtained in these cases is presented in Table

**Table I.** Integrable Cases of Eq. (4.1) and the Associated Constants of Motion

| Parameters | Invariant $I$ |
|---|---|
| $\Omega = \omega = \alpha/3$ | $\alpha\Psi\phi(\phi^2 - \Psi^2)/3 + \Psi_t\phi_t$ |
| $\Omega = \omega = \alpha$ | $\Psi_t\phi - \phi_t\Psi$ |
| $\Omega = 8\alpha/3,\ \omega = \alpha/6$ | $\alpha\Psi\phi^4 - 6\Psi\phi_t^2 - 2\alpha\Psi^3\phi^2 + 6\phi\Psi_t\phi_t$ |
| $\Omega = \alpha/6,\ \omega = 8\alpha/3$ | $6\Psi\Psi_t\phi + 2\alpha\Psi^2\phi^3 - 6\Psi_t^2\phi - \alpha\Psi^4\phi$ |
| $\Omega = 8\alpha/3,\ \omega = \alpha/3$ | $\alpha^2\Psi^4\phi^4 + 12\alpha\Psi\phi^3\Psi_t\phi_t + 9\phi_t^4 - 18\alpha\Psi^2\phi^2\phi_t^2 + \alpha^2\phi^8/4 - 3\alpha\phi^4\Psi_t^2 - \alpha^2\Psi^2\phi^6$ |
| $\Omega = \alpha/3,\ \omega = 8\alpha/3$ | $\alpha^2\Psi^4\phi^4 - 12\alpha\Psi^3\phi\Psi_t\phi_t + 9\Psi_t^4 - 3\alpha\Psi^4\Psi_t^2 + 18\alpha\Psi^2\phi^2\Psi_t^2 + \alpha^2\Psi^8/4 + 3\alpha\Psi^4\phi_t^2 - \alpha^2\phi^2\Psi^6$ |

**Table II.** Third Invariant $J$ for the Integrable Cases with $I = 0$

| Parameters | Invariant $J$ |
|---|---|
| $\Omega = \omega = \alpha/3$ | $3\Psi_t^2/\phi^2 + 2\alpha\Psi^2$ |
| $\Omega = \omega = \alpha$ | $\Psi/\phi$ |
| $\Omega = 8\alpha/3,\ \omega = \alpha/6$ | $\Psi(2\Psi^2 - \phi^2)/\phi^3\phi_t$ |
| $\Omega = \alpha/6,\ \omega = 8\alpha/3$ | $\phi(2\phi^2 - \Psi^2)/\Psi^3\Psi_t$ |
| $\Omega = 8\alpha/3,\ \omega = \alpha/3$ | $4\alpha\Psi^4 - 4\alpha\Psi^2\phi^2 + \alpha\Psi^4 - 3\Psi_t^2$ $\times (\Psi(4\alpha\Psi^4 - 2\alpha\Psi^2\Phi^2 - 3\Psi_t^2))^{-1}$ |
| $\Omega = \alpha/3,\ \omega = 8\alpha/3$ | $4\alpha\phi^4 - 4\alpha\Psi^2\phi^2 + \alpha\Psi^4 - 3\phi_t^2$ $\times (\phi(4\alpha\phi^4 - 2\alpha\Psi^2\Phi^2 - 3\phi_t^2))^{-1}$ |

II and it is a nonpolynomial function. The peculiarity of the value $I = 0$ can be appreciated if one uses the variables $H$, $\psi$, and $\phi$ and eliminates $\dot{\psi}$ using the energy constraint (3.5). In these variables, all the invariants assume a special form reported in Table III; they all factorize as

$$I = \phi^n \mathcal{F}(H, \psi) \tag{4.3}$$

and since $\phi \neq 0$ it must be that $\mathcal{F}(H, \psi) = 0$ for $I = 0$. This relation can then be used to eliminate $H$ or $\psi$ from Eq. (3.9) or (3.10), and the scale factor is determined by integrating Eq. (3.6).

Whether one integrates the equations in the rescaled variables $\Psi$ and $\phi$ by using the invariants $E = 0$, $I = 0$, and $J$, or one does it in the variables $H$ and $\psi$ using $I = 0$ as given in Table III, is only a matter of convenience and simplicity, since the results obtained in both ways coincide. In each case a judicious choice leads to a huge simplification in the amount of calculations needed to obtain the final expressions.

**Table III.** Same as Table I, But Using the Variables $H$, $\psi$, and $\phi$

| Parameters | Invariant $I$ |
|---|---|
| $\Omega = \omega = \alpha/3$ | $-(\sqrt{6\kappa}/216)\phi^4(2\alpha\kappa\psi^3 - 12\alpha\psi - H\sqrt{216\alpha + 216\kappa H^2 - 216\alpha\kappa\psi^2 + 6\alpha\kappa^2\psi^4})$ |
| $\Omega = \omega = \alpha$ | $-(\phi^3/12)(2\kappa H\psi - \sqrt{1728\alpha + 216\kappa H^2 - 216\alpha\kappa\psi^2 + 3\alpha\kappa^2\psi^4})$ |
| $\Omega = 8\alpha/3,\ \omega = \alpha/6$ | $-(\sqrt{6\kappa}/18)\phi^5(3\kappa\psi H^2 - 3\alpha\psi + \alpha\kappa\psi^3 - H\sqrt{27\alpha - 54\alpha\kappa\psi^2 + 12\alpha\kappa^2\psi^4 + 54\kappa H^2})$ |
| $\Omega = \alpha/6,\ \omega = 8\alpha/3$ | $-(\phi^5/72)(576\alpha - 96\alpha\kappa\psi^2 + 72\kappa H^2 + 3\alpha\kappa^2\psi^4 - 2\kappa\psi H\sqrt{1728\alpha + 216\kappa H^2 - 216\alpha\kappa\psi^2 + 3\alpha\kappa^2\psi^4})$ |
| $\Omega = 8\alpha/3,\ \omega = \alpha/3$ | $(\phi^8/36)(9\kappa^2H^4 - 3\alpha^2\kappa^2\psi^4 - 18\kappa^2\alpha H^2\psi^2 - 9\alpha^2 + 12\kappa\alpha^2\psi^2 + 4\kappa\alpha H\psi\sqrt{54\alpha + 54\kappa H^2 - 54\kappa\alpha\psi^2 + 12\kappa^2\alpha\psi^4})$ |
| $\Omega = \alpha/3,\ \omega = 8\alpha/3$ | $(\phi^8/216)(54\kappa^2H^4 - 48\kappa^2\alpha^2\psi^4 + 3\kappa^3\alpha H^2\psi^4 + 864\kappa\alpha H^2 + 2\alpha^2\kappa^3\psi^6 + 3456\alpha^2 - 2\kappa^2\alpha H\psi^3\sqrt{216\kappa H^2 - 216\kappa\alpha\psi^2 + 6\kappa^2\alpha\psi^4 + 1728\alpha})$ |

We note that in this approach we make no assumption on the equation of state governing the field $\psi$, which is obtained in a self-consistent way from the solutions. As discussed in a forthcoming publication, not every type of equation of state is allowed in this model and all the linear equations of state allowed can be obtained.

Before presenting the analytic solutions it is worth noting that since the original system is Hamiltonian and therefore reversible, all the solutions have a corresponding time-reversed solution. This means that if $(\psi(\tau), \alpha(\tau), H(\tau))$ is a solution of our system, then $(\psi(-\tau), \alpha(-\tau), -H(-\tau))$ is also a solution of the same system. For brevity, only one of the possibilities (when there are multiple ones) is presented. In addition, the initial conditions are chosen, without loss of generality, in such a way that the expressions for the solutions are simplified. A different choice of initial conditions is equivalent to a shift of the origin of time.

We proceed to present all the solutions obtained for the six integrable cases of our system:

*Case 1.* $\omega = \alpha/6$, $\Omega = 8\alpha/3$.

By using the invariant $I$ in Table III and setting $I = 0$ one obtains

$$3\kappa\psi H^2 - 3\alpha\psi + \alpha\kappa\psi^3 - \sqrt{54\kappa H^2 - 54\alpha\kappa\psi^2 + 12\alpha\kappa^2\psi^4 + 27\alpha H} = 0 \tag{4.4}$$

and the solution for $\psi$ is

$$\psi = \pm\sqrt{\frac{3}{\alpha}}\,H \tag{4.5}$$

or

$$\psi = \pm\sqrt{\frac{3}{2\alpha\kappa}}\,[2\alpha + \kappa H^2 \pm H\sqrt{\kappa^2 H^2 - 4\alpha\kappa}]^{1/2} \tag{4.6}$$

The two choices of sign in Eq. (4.6) are independent of each other, i.e., there are four different branches. Each branch must be separately checked to avoid spurious possibilities. In fact, only the positive sign in the second symbol $\pm$ of Eq. (4.6) leads to a valid solution, while both signs are valid for the first symbol $\pm$. In each case below different branches occur and must be similarly checked. Equation (4.6) can be substituted into Eq. (3.10) and after some algebra the following solution is obtained:

$$\psi(\tau) = \pm\sqrt{\frac{6}{\kappa}}\,\frac{1}{\cos(\sqrt{2\alpha/\kappa}\tau)} \tag{4.7}$$

$$a(\tau) = a_0\,\frac{\sqrt{|\sin(\sqrt{2\alpha/\kappa}\tau)|}}{|\cos(\sqrt{2\alpha/\kappa}\tau)|} \tag{4.8}$$

where here and in the following, $a_0$ is an arbitrary integration constant. The Hubble function follows from Eqs. (3.6) and (4.8). Plots of $\psi(\tau)$, $\phi(\tau)$, and $H(\tau)$ are shown in Fig. 1, in which only the positive sign for the scalar field (4.7) is shown. The same choice of sign applies to the rest of the figures.

The energy density and pressure are obtained from Eqs. (3.7) and (3.8),

$$\sigma(\tau) = \frac{3\alpha}{2\kappa^2}\frac{3v - 4}{v(v - 1)} \tag{4.9}$$

$$p(\tau) = -\frac{\alpha}{2\kappa^2}\frac{21v - 20}{v(v - 1)} \tag{4.10}$$

with $v(\tau) \equiv \cos^2(\sqrt{2\alpha/\kappa}\tau)$; the equation of state is

$$p = \gamma(\tau)\sigma \tag{4.11}$$

where

$$\gamma(\tau) = -\frac{21v - 20}{3(3v - 4)} \tag{4.12}$$

Equation (4.11) is linear and, for $\tau = 0$, it reduces to the radiation equation of state $p = \sigma/3$.

The other possibility associated with Eq. (4.5) leads to the solution

$$\psi(\tau) = \pm\frac{3}{\sqrt{\kappa}}\tanh\left(\sqrt{\frac{\alpha}{\kappa}}\,\tau\right) \tag{4.13}$$

$$a(\tau) = a_0\,\mathrm{sech}\left(\sqrt{\frac{\alpha}{\kappa}}\,\tau\right) \tag{4.14}$$

$$\sigma(\tau) = \frac{3\alpha}{\kappa^2}\tanh^2\left(\sqrt{\frac{\alpha}{\kappa}}\,\tau\right) + \frac{3\alpha}{2\kappa^2} \tag{4.15}$$

and the equation of state is

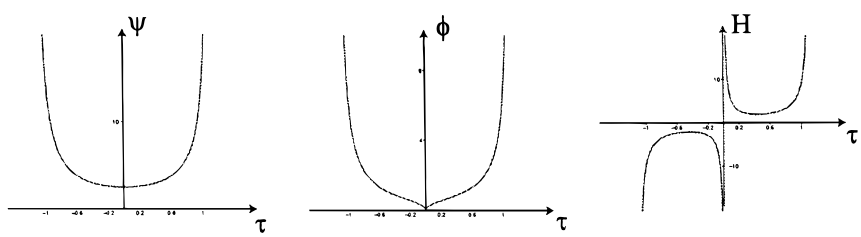


**Fig. 1.** $\psi$, $\phi$, and $H$ as functions of the proper time $\tau$, as given by Eqs. (4.7) and (4.8) for $\omega = \alpha/6$ and $\Omega = 8\alpha/3$.

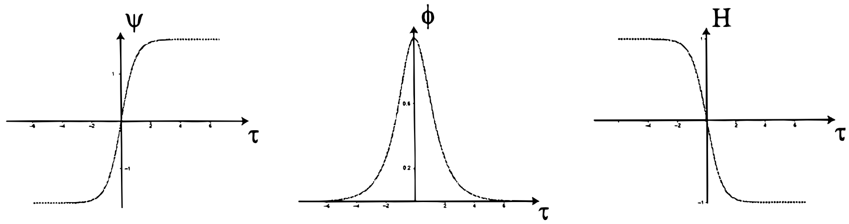


**Fig. 2.** $\psi$, $\phi$, and $H$ as functions of the proper time $\tau$, as given by Eqs. (4.13) and (4.14) for $\omega = \alpha/6$ and $\Omega = 8\alpha/3$.

$$p = -\frac{5}{3}\,\sigma + \frac{3\alpha}{\kappa^2} \tag{4.16}$$

The corresponding plots are given in Fig. 2. This solution begins as an expanding de Sitter universe with $p = -\sigma$ as $\tau \to -\infty$, goes through a radiation phase with $p = \sigma/3$ at $\tau = 0$, and then evolves into a contracting de Sitter space as $\tau \to +\infty$.

*Case 2.* $\omega = 8\alpha/3$, $\Omega = \alpha/6$.

By proceeding analogously to the previous case, one obtains the solutions

$$\psi(\tau) = \pm\sqrt{(12/\kappa)}[1 + \cos(\sqrt{8\alpha/\kappa}\,\tau)]^{-1/2} \tag{4.17}$$

$$a(\tau) = a_0\sqrt{|\sin(\sqrt{8\alpha/\kappa}\,\tau)|}\sqrt{1 + |\cos(\sqrt{8\alpha/\kappa}\,\tau)|} \tag{4.18}$$

$$\sigma(\tau) = \frac{6\alpha}{\kappa^2}\,\frac{4v - 5}{v^2 - 1} \tag{4.19}$$

$$p(\tau) = -\frac{2\alpha}{\kappa^2}\,\frac{8v - 7}{v^2 - 1} \tag{4.20}$$

with $v(\tau) \equiv \cos(\sqrt{8\alpha/\kappa}\,\tau)$. The equation of state is given by Eq. (4.11) with

$$\gamma(\tau) = -\frac{8v - 7}{3(4v - 5)} \tag{4.21}$$

Again, $p \to \sigma/3$ as $\tau \to 0$; plots are given in Fig. 3.

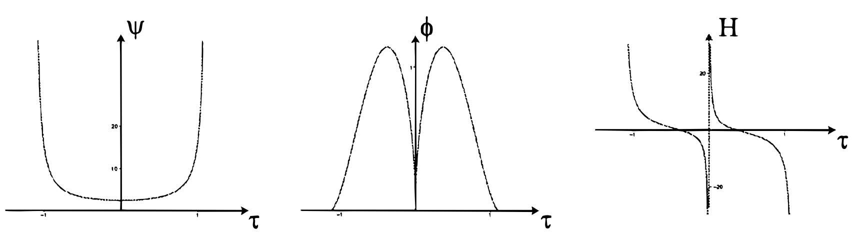


**Fig. 3.** $\psi$, $\phi$, and $H$ as functions of proper time $\tau$, as given by Eqs. (4.17) and (4.18) for $\omega = 8\alpha/3$ and $\Omega = \alpha/6$.

*Case 3.* $\omega = \alpha/3$, $\Omega = 8\alpha/3$.

After cumbersome computations, two solutions are obtained; the first one is

$$H(\tau) = \sqrt{\frac{\alpha}{5\kappa}}\,\frac{(u + 1 + 8\sqrt{u})}{(u - 1)} \tag{4.22}$$

where $u \equiv \exp(-2\sqrt{5\alpha/\kappa}\tau)$; the scalar field and the scale factor are

$$\psi(\tau) = \pm\frac{1}{\sqrt{3\alpha\kappa}}\,[3\alpha + 5\kappa H^2 + \sqrt{3\alpha\kappa H^2 + \kappa^2 H^4}]^{1/2} \tag{4.23}$$

and

$$a(\tau) = a_0 \exp\left\{\frac{1}{10}\left[16\tanh^{-1}(\sqrt{u}) - \ln\frac{(u - 1)^2}{u}\right]\right\} \tag{4.24}$$

The corresponding plots appear in Fig. 4; this solution begins as a contracting universe that evolves into a singular ($a = 0$) de Sitter solution as $\tau \to +\infty$. The time-reversed solution begins with a singularity as $\tau \to -\infty$, with constant Hubble function.

The second solution is

$$\psi(\tau) = \pm\sqrt{\frac{3}{\kappa}}\,\frac{1}{\cos(\sqrt{(\alpha/\kappa)}\tau)} \tag{4.25}$$

$$a(\tau) = \frac{a_0}{\cos(\sqrt{(\alpha/\kappa)}\tau)} \tag{4.26}$$

$$\sigma(\tau) = \frac{3\alpha}{\kappa^2}\,\frac{1}{\cos^2(\sqrt{(\alpha/\kappa)}\tau)} \tag{4.27}$$

with the equation of state

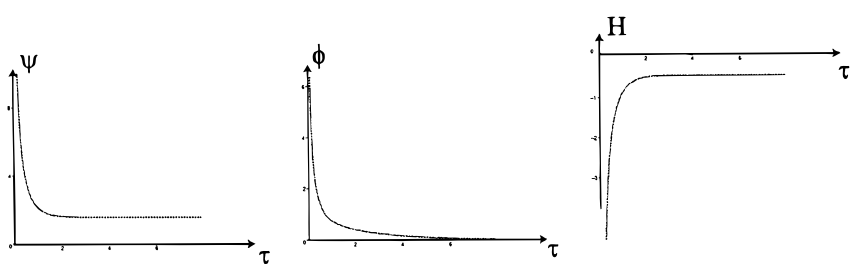


**Fig. 4.** $\psi$, $\phi$, and $H$ as functions of the proper time $\tau$, as given by Eqs. (4.23) and (4.24) for $\omega = \alpha/3$ and $\Omega = 8\alpha/3$.

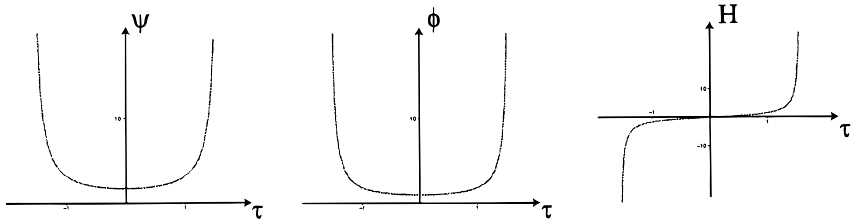


**Fig. 5.** $\psi$, $\phi$, and $H$ as functions of the proper time $\tau$, as given by Eqs. (4.25) and (4.26) for $\omega = \alpha/3$ and $\Omega = 8\alpha/3$.

$$p = -\frac{5}{3}\sigma \tag{4.28}$$

Plots are presented in Fig. 5.

*Case 4.* $\omega = 8\alpha/3$, $\Omega = \alpha/3$.
In this case the only solution obtained is

$$\psi(\tau) = \pm 2\sqrt{\frac{3}{\kappa}}\frac{1}{\cos(\sqrt{(2\alpha/\kappa)}\tau)} \tag{4.29}$$

$$a(\tau) = a_0 \cos^2(\sqrt{(2\alpha/\kappa)}\tau) \tag{4.30}$$

$$\sigma(\tau) = \frac{24\alpha}{\kappa^2}\frac{1}{\cos^2(\sqrt{(2\alpha/\kappa)}\tau)} \tag{4.31}$$

and the equation of state is

$$p = -\frac{2}{3}\sigma \tag{4.32}$$

with plots presented in Fig. 6.

*Case 5.* $\omega = \Omega = \alpha/3$.
The solution is

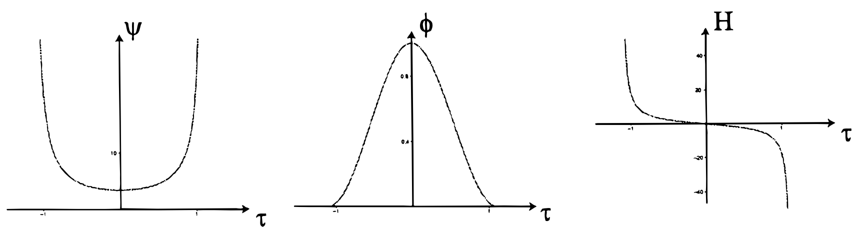


**Fig. 6.** $\psi$, $\phi$, and $H$ as functions of the proper time $\tau$, as given by Eqs. (4.29) and (4.30) for $\omega = 8\alpha/3$ and $\Omega = \alpha/3$.

$$\psi = \pm\sqrt{\frac{6}{\kappa}}\tan\left(\sqrt{\frac{\alpha}{\kappa}}\,\tau\right) \tag{4.33}$$

$$a(\tau) = a_0\cos^2\left(\sqrt{\frac{\alpha}{\kappa}}\,\tau\right) \tag{4.34}$$

$$\sigma(\tau) = \frac{4\alpha}{3\kappa^2}\tan^2\left(\sqrt{\frac{\alpha}{\kappa}}\,\tau\right) + \frac{3\alpha}{\kappa^2} \tag{4.35}$$

and the equation of state is

$$p = -\frac{2}{3}\sigma + \frac{3\alpha}{\kappa^2} \tag{4.36}$$

Plots corresponding to this solution are given in Fig. 7.

*Case 6.* $\omega = \Omega = \alpha$.

In this case all the solutions for $I = 0$ are de Sitter universes, as discussed in ref. 1, and correspond to degenerate fixed points of Eqs. (3.9) and (3.10).

## 5. CONCLUDING REMARKS

Among the above solutions, those represented in Figs. 2 and 4 are worth special attention. The former corresponds to a heteroclinic orbit (i.e., an orbit connecting two fixed points of the differential system) in the phase space, with a linear equation of state; it begins from an expanding ($H > 0$) and evolves to a contracting ($H < 0$) de Sitter universe. It is easily seen from Eqs. (4.15) and (4.16) that at the time $\tau = 0$ one has the radiation equation of state $p(0) = \sigma(0)/3$. Furthermore, as $\tau \to \pm\infty$ one obtains the de Sitter equation of state $p(\pm\infty) + \sigma(\pm\infty) = 0$, with $\sigma(\pm\infty) = 9\alpha/2\kappa^2$.

The second solution (Fig. 4) describes an evolution starting from a singularity ($H = -\infty$) at $\tau = 0$ and evolving to a contracting de Sitter spacetime. We stress that, in contrast with the heteroclinic orbit presented in

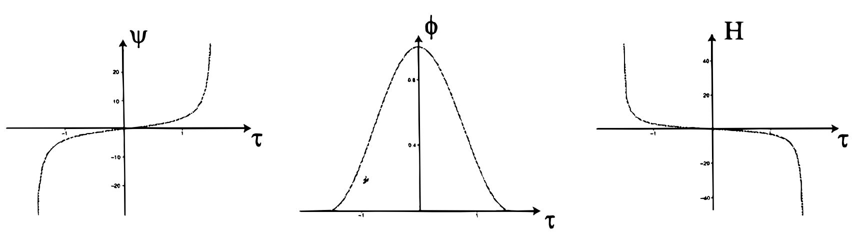


**Fig. 7.** $\psi$, $\phi$, and $H$ as functions of the proper time $\tau$, as given by Eqs. (4.33) and (4.34) for $\omega = \Omega = \alpha/3$.

ref. 2, which represents a so-called critical solution, the solution (4.13) and (4.14) is only valid in the presence of a nonvanishing cosmological constant $\omega$ (or $\Lambda$). Another peculiarity is the special character of the linear equation of state obtained for certain solutions for which the pressure is not an affine function of the energy density, which yields as asymptotic cases the more familiar de Sitter and radiative equations of state. We refer the reader to refs. 1 and 2 for details of this cosmological model.

## ACKNOWLEDGMENTS

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